Splitting cubic circle graphs

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Abstract

We show that every 3-regular circle graph has at least two pairs of twin vertices; consequently no such graph is prime with respect to the split decomposition. We also deduce that up to isomorphism, K_4 and $K_{3,3}$ are the only 3-connected, 3-regular circle graphs.

Keywords. circle graph, split decomposition, regular graph

Mathematics Subject Classification. 05C62

1 Introduction

Circle graphs have been introduced several times, in several contexts. The intersection graph of a family of chords in a circle seems to have first been mentioned in print by Zelinka [17]; he gave credit for the idea to Kotzig, whose seminal work [13] founded the special theory of 4-regular graphs. Brahana's separation matrix [4] – in essence, the adjacency matrix of a circle graph – was introduced decades earlier, in connection with the geometry of surfaces. Circle graphs achieved broad recognition in the 1970s, when Even and Itai [8] considered circle graphs in relation to the analysis of permutations using stack and queues; Bouchet [1] and Read and Rosenstiehl [15] discussed the interlacement graphs of double occurrence words in connection with the famous Gauss problem of characterizing generic curves in the plane; and Cohn and Lempel [5] related the cycle structure of a certain kind of permutation to the $GF(2)$-nullity of an associated link relation matrix (which is also the adjacency matrix of a circle graph). An account of the early combinatorial theory appears in Golumbic's classic book [12].

Definition 1 *Let $W = w_1 ... w_{2n}$ be a double occurrence word, i.e., a sequence in which n letters appear, each letter appearing twice. Then the* interlacement graph *$\mathcal{I}(W)$ is a graph with n vertices, labeled by the letters appearing in W. Two vertices a and b of $\mathcal{I}(W)$ are adjacent if and only if the corresponding letters appear in W in the order $abab$ or $baba$. A simple graph that can be realized as an interlacement graph of a double occurrence word is a* circle graph.

During the last forty years the theory of circle graphs has been sharpened considerably. Polynomial-time recognition algorithms were developed before the new millennium by Bouchet [2], Naji [14] and Spinrad [16]. More recently, Courcelle [6] has observed that circle graphs are well described in the framework of monadic second-order logic, and Gioan, Paul, Tedder and Corneil have provided the first subquadratic recognition algorithm [10, 11].

The crucial tool used to design recognition algorithms for circle graphs is the split decomposition of Cunningham [7]. We recall only the basic definition here, and defer to the literature ([6, 7, 11] for instance) for thorough explanations of this important idea.

Definition 2 *Let G be a simple graph. A* split $(V_1,\ X_1;\ V_2,\ X_2)$ *of G is given by a partition $V(G) = V_1 \cup V_2$ with $|V_1|,\ |V_2| \geq 2$ and subsets $X_i \subseteq V_i$ with these properties: the complete bipartite graph with vertex-classes X_1 and X_2 is a subgraph of G, and G does not have any other edge from V_1 to V_2.*

Every simple graph of order 4 has a split. A graph with five or more vertices that has no split is said to be *prime*.

Three types of splits are particularly simple. Let G be a graph with four or more vertices.

- Suppose v_1 and v_2 are twin vertices of G, i.e., they have the same neighbors outside $\{v_1, v_2\}$. Then G has a split with $V_1 = X_1 = \{v_1, v_2\}$.
- Suppose G is not connected. Let H be a union of some but not all connected components of G, such that H includes at least two vertices. If $h \in V(H)$ then G has a split with $V_1 = V(H)$ and $X_1 = \varnothing$ or G has a split with $V_1 = V(H - h)$ and $X_2 = \{h\}$.
- Suppose G has a cutpoint v. Let H be a union of some but not all connected components of $G - v$, such that H includes at least two vertices. Then G has a split with $V_1 = V(H)$ and $X_2 = \{v\}$.

A fundamental part of the theory of circle graphs and split decompositions is the following operation, which is motivated by the properties of double occurrence words [1, 13, 15]. We use $N(v)$ to denote the open neighborhood of v in G, i.e., the set of vertices $w \neq v$ such that $vw \in E(G)$.

Definition 3 *Let v be a vertex of a simple graph G. Then the* local complement *of G with respect to v is the graph G^v with $V(G^v) = V(G)$ and $E(G^v) = \{xy \mid$ either $x \notin N(v)$ and $xy \in E(G)$ or $x, y \in N(v)$ and $xy \notin E(G)\}$.*

In some references this operation is called *simple* local complementation, to distinguish it from a related operation that involves looped vertices. We consider only simple graphs in this paper, so we need not be so careful here. Two important properties of local complementation are that G is a circle graph if and only if G^v is a circle graph, and that G has a split $(V_1,\ X_1;\ V_2,\ X_2)$ if and only if G^v has a split $(V_1,\ Y_1;\ V_2,\ Y_2)$.

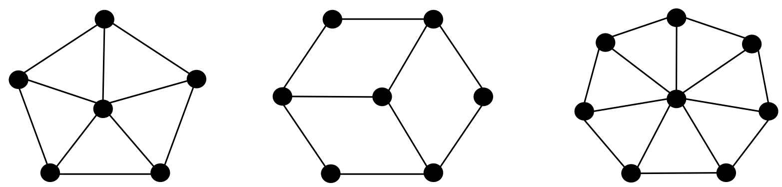

Figure 1: Bouchet's obstructions: W_5, BW_3 and W_7.

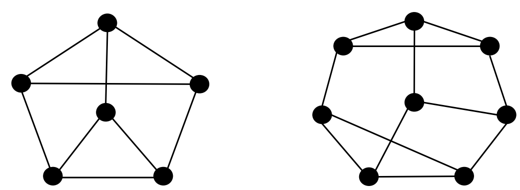

Figure 2: Prime, 3-connected cubic graphs locally equivalent to W_5 and W_7.

A central result involving circle graphs is Bouchet's characterization by obstructions [3]. Recall that *local equivalence* is the equivalence relation generated by isomorphisms and local complementations.

Theorem 4 *[3] A simple graph G is a circle graph if and only if no graph locally equivalent to G has one of the graphs of Figure 1 as an induced subgraph.*

Observe that W_5 and W_7 are locally equivalent to the graphs pictured in Figure 2, both of which are prime, 3-connected, and 3-regular. The purpose of this paper is to prove that these local equivalences are no mere coincidence:

Theorem 5 *Let G be a 3-regular circle graph. Then G has at least two disjoint pairs of twin vertices.*

Theorem 5 immediately implies the following.

Corollary 6 *Let G be a 3-regular circle graph. Then G is not prime.*

In Section 4 we deduce another consequence of Theorem 5.

Corollary 7 *Let G be a 3-regular circle graph, which is not isomorphic to K_4 or $K_{3,3}$. Then G is not 3-connected.*

Before proceeding we should take a moment to thank Robert Brijder for the many inspirations provided by our long correspondence and collaboration.

2 Three lemmas

In this section we recall three elementary results about double occurrence words and circle graphs.

Lemma 8 *If G is a circle graph with a vertex v then $G - v$ and G^v are also circle graphs.*

Proof. For $G-v$, take a double occurrence word whose interlacement graph is G and remove the two occurrences of v. For G^v, take a double occurrence word whose interlacement graph is G and reverse the subword between the two occurrences of v. ■

Lemma 9 *If W' is obtained from a double occurrence word W by some sequence of cyclic permutations*

$$w_1...w_{2n} \mapsto w_i w_{i+1}...w_{2n} w_1..w_{i-1}$$

and reversals

$$w_1...w_{2n} \mapsto w_{2n} w_{2n-1}...w_2 w_1$$

then $\mathcal{I}(W) = \mathcal{I}(W')$.

Double occurrence words related as in Lemma 9 are said to be *cyclically equivalent*. By the way, the converse of Lemma 9 is false in general; a complete characterization of double occurrence words with the same connected interlacement graph has been provided by Ghier [9].

Lemma 10 *Let G_1 and G_2 be disjoint simple graphs, and let G be a graph obtained by attaching G_1 to G_2 with a single edge. Then G is a circle graph if and only if both G_1 and G_2 are circle graphs.*

Proof. If G is a circle graph, it yields G_1 and G_2 through vertex deletion.

For the converse, suppose $W_1 = w_1...w_{2a}$ and $W_2 = x_1...x_{2b}$ are double occurrence words whose interlacement graphs are G_1 and G_2, respectively. After cyclic permutation, we may presume that the one additional edge of G attaches w_{2a} to x_1. Then the word

$$W = w_1...w_{2a-1} x_1 w_{2a} x_2...x_{2b}$$

has G as its interlacement graph. ■

3 Proof of Theorem 5

Before beginning the proof, we should mention that we sometimes say "two pairs" rather than "two disjoint pairs" while discussing Theorem 5. In fact, the theorem is equivalent to the weaker-seeming assertion that every cubic circle graph has two pairs of twin vertices. For if a cubic graph has two intersecting

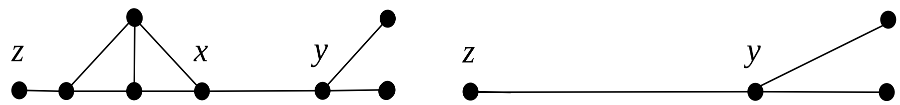

Figure 3: If G has more than two cutpoints, its minimality is contradicted.

pairs of adjacent twins, then the three twin vertices and their shared neighbor constitute a 4-clique, which must be a whole connected component; the 4-clique provides three disjoint pairs of twin vertices. If a cubic graph has two intersecting pairs of nonadjacent twins then the three twin vertices and their three neighbors constitute a 3,3-biclique, which must again be a whole connected component; the biclique provides nine disjoint pairs of twin vertices. (An easy argument shows that a pair of adjacent twins cannot intersect a pair of nonadjacent twins in any graph.)

3.1 A minimal counterexample is 2-connected

Let G be a minimal counterexample to Theorem 5, i.e., a cubic circle graph that does not have two disjoint pairs of twin vertices, with the smallest possible number of vertices. Then G must certainly be connected, for if not then each connected component of G is itself a smaller counterexample.

Proposition 11 *Let G be a cubic circle graph that does not have two pairs of twin vertices, and is of the smallest order for such a graph. If G is not 2-connected then G has precisely two cutpoints.*

Proof. Suppose G has a cutpoint, x. As x is of degree 3, one of the components of $G - x$ is connected to x by only one edge, e. Then e is an isthmus, and its other end-vertex is a cutpoint; denote the other end-vertex y.

Suppose G has a third cutpoint, z. Interchanging the labels of x and y if necessary, we may presume that x and z are vertices of the same component of $G - e$; and interchanging the labels of z and a neighbor of z, we may presume that there is an isthmus between x and z. (A portion of an example is indicated on the left in Figure 3.) Let G' be the cubic graph obtained from G by removing all vertices between y and z, and then attaching y to z by an edge, as indicated on the right in Figure 3. Lemma 10 tells us that G' is a circle graph. Moreover, it is clear that every pair of twins in G' is also a pair of twins in G, so G' does not have two pairs of twin vertices. But this contradicts the minimality of G. ■

Proposition 12 *Let G be a cubic circle graph that does not have two pairs of twin vertices, and is of the smallest order for such a graph. Suppose G has two cutpoints, x and y. Then at least one of x, y does not appear on a 3-circuit of G.*

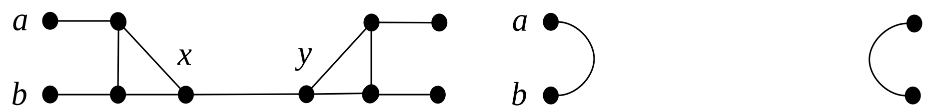

Figure 4: G, G_x and G_y.

Proof. Suppose instead that x and y both appear on 3-circuits of G. If the two other vertices of the 3-circuit containing x are twins then their other neighbor is a cutpoint, as illustrated on the left in Figure 3. But G does not have a third cutpoint, so these two vertices must not be twins. The same argument shows that the other two vertices of the 3-circuit containing y are not twins, so the situation in G is as indicated on the left-hand side of Figure 4. As indicated on the right-hand side of the figure, we obtain two smaller cubic circle graphs by deleting x and y and then performing local complementations and deletions at the four resulting degree-2 vertices. Call the two smaller graphs G_x and G_y. The minimality of G implies that each of G_x and G_y has two pairs of twin vertices.

As G does not have two pairs of twin vertices, it must be the case that for at least one of G_x and G_y, no pair of twin vertices is also a pair of twins in G. We assume that no pair of twin vertices from G_x is also a twin pair in G. Then there are two pairs of twins in G_x whose twin relationship is "disrupted" in G.

Such disruption can only occur if each pair of twins includes one of the vertices of G_x denoted a and b in Figure 4, because these are the only vertices of G_x whose neighborhoods in G and G_x are not the same. Let a' and b' be vertices of G that are twins of a and b in G_x.

Suppose a and a' are adjacent twins in G_x. If b and b' are adjacent too, then $N_G(a') = \{a, b, b'\}$ and $N_G(b') = \{a, a', b\}$, so a' and b' are adjacent twins in G. This is a contradiction, so b and b' are not adjacent. Consequently there is a vertex z such that $N_{G_x}(b) = N_{G_x}(b') = \{a, a', z\}$; but then z is a cutpoint of G, which separates $\{a, a', b, b'\}$ from the third neighbor of z. We conclude that a and a' are nonadjacent twins in G_x; the same argument shows that b and b' are nonadjacent twins in G_x.

We claim that Figure 5 accurately reflects the situations in G_x and G. To verify the claim, note first that the vertices labeled c and d in Figure 5 must be distinct, as G has no cutpoint other than x and y. These vertices must also be nonadjacent, because their being adjacent would imply that d is a twin of a' and c is a twin of b', contradicting the hypothesis that G does not have two pairs of twins. Furthermore, c and d cannot share a neighbor, because a shared neighbor would be a new cutpoint of G. These three observations justify the claim.

Now, notice that Figure 6 indicates that the same sort of twin disruption occurs in a cubic graph G' that is smaller than G. More generally, it is clear that G and G' have precisely the same pairs of twin vertices. The minimality

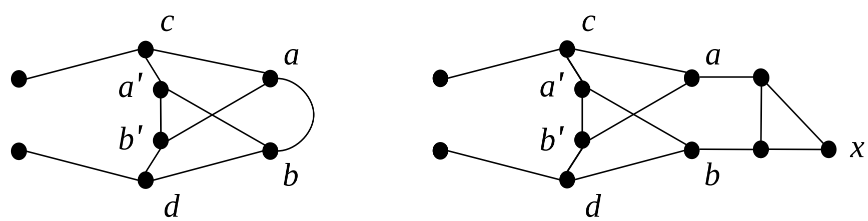

Figure 5: Two pairs of twins in G_x are not twins in G.

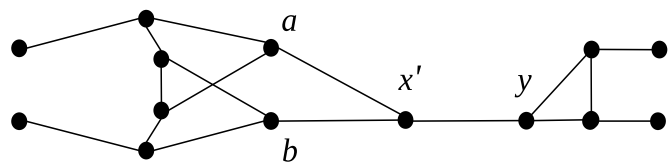

Figure 6: Two pairs of twins in G_x are not twins in G'.

of G requires that G' not be a circle graph.

However, we can obtain G' from G as follows. Let C_x and C_y be the connected components of $G - xy$. Lemma 10 tells us that both are circle graphs. Observe that $C_x - x$ has two vertices of degree 2, the neighbors of x in C_x. Denote these neighbors x' and x'', take the local complement of $C_x - x$ with respect to x'', and then remove x''. Denote the resulting graph C'_x. We obtain G' from C'_x and C_y by attaching x' to y with an edge, so Lemma 10 tells us that G' is a circle graph. ■

Proposition 13 *Let G be a cubic circle graph that does not have two pairs of twin vertices, and is of the smallest order for such a graph. Then G has no cutpoint.*

Proof. Suppose instead that such a G has a cutpoint, x. According to Propositions 11 and 12, x has a neighbor y that is the only other cutpoint of G, and we may presume that x does not appear on a 3-circuit of G. Consequently G must fall into one of the two cases pictured on the left-hand side of Figure 7. Let G_x and G_y be the two smaller graphs pictured on the right-hand side of Figure 7. Each of G_x, G_y is obtained from G using vertex deletions and local complementations, so each of G_x, G_y is a circle graph; the minimality of G implies that each of G_x, G_y has two pairs of twin vertices.

Most of the proof consists of a verification of the following.

Claim. It cannot be that no twins of G_x are twins in G.

Suppose the claim is false. As in the proof of Proposition 12, this requires that the two neighbors of x in G_x appear in two disjoint twin pairs, which are

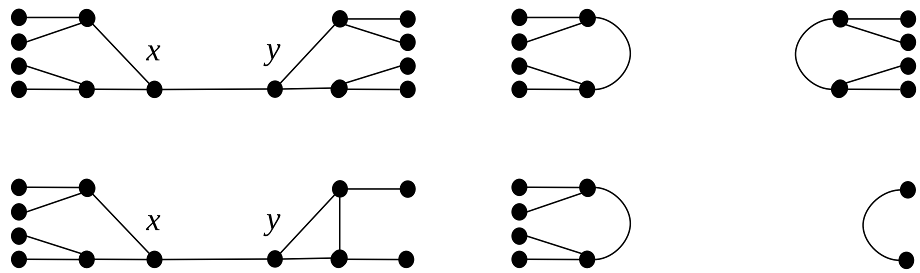

Figure 7: The two cases of Proposition 13.

disrupted in G. Let the neighbors be denoted a and b, and suppose a' and b' are twins of a and b in G_x.

Suppose a and a' are adjacent twins in G_x. If b and b' are adjacent too, then $N_G(a') = \{a, b, b'\}$ and $N_G(b') = \{a, a', b\}$, so a' and b' are adjacent twins in G. This contradiction tells us that b and b' are not adjacent, so there is a vertex z such that $N_{G_x}(b) = N_{G_x}(b') = \{a, a', z\}$. Then z is a cutpoint of G, which separates $\{a, a', b, b', x\}$ from the rest of G_x. We see that a and a' cannot be adjacent; the same argument shows that b and b' are nonadjacent too.

There are two configurations to consider, indicated in Figure 8. In both configurations, there must be no twins in the portion of G_x that is not pictured in Figure 8. To see why one of these configurations must apply, notice that a' and b' cannot share a neighbor, because a shared neighbor would be a cutpoint. Consequently the two unlabeled vertices of Figure 8 are indeed distinct. If these two vertices were adjacent, they would be twins of a' and b' in G. And if these two vertices were to share a neighbor, then that neighbor would be a cutpoint.

Consider the first configuration, indicated on the left in Figure 8. Let G' be the graph obtained from G by removing a, a', b, b' and the two unlabeled vertices that appear in the figure, and then inserting the edges cx and dx. Then G' can also be obtained from G as follows: First, remove a' and b'. Then, perform a local complementation followed by a vertex deletion at a, b, and each of the two unlabeled vertices. It follows that G' is a cubic circle graph. The minimality of G requires that G' have at least one pair of twin vertices that are not twins in G; this pair can only be $\{c, d\}$. But then the unpictured third neighbors of c and d are the same, and this vertex is a cutpoint of G.

The second configuration resembles the first, but c and d are not neighbors. Consider the smaller graph obtained from G_x by first deleting the vertices a' and b', then taking the local complement with respect to each of the four resulting degree-2 vertices in turn, and deleting it. This smaller graph must have two pairs of twin vertices, and both of these twin pairs must be disrupted in G. This can only occur if each pair of twins includes one of c, d. That is, the smaller graph must be as pictured at the top of Figure 9. The other information in Figure 9 follows from our hypotheses: the unlabeled neighbors of c' and d' must be

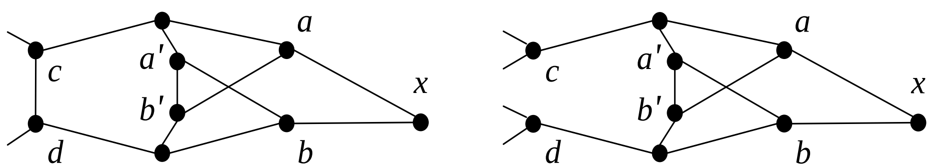

Figure 8: Two configurations.

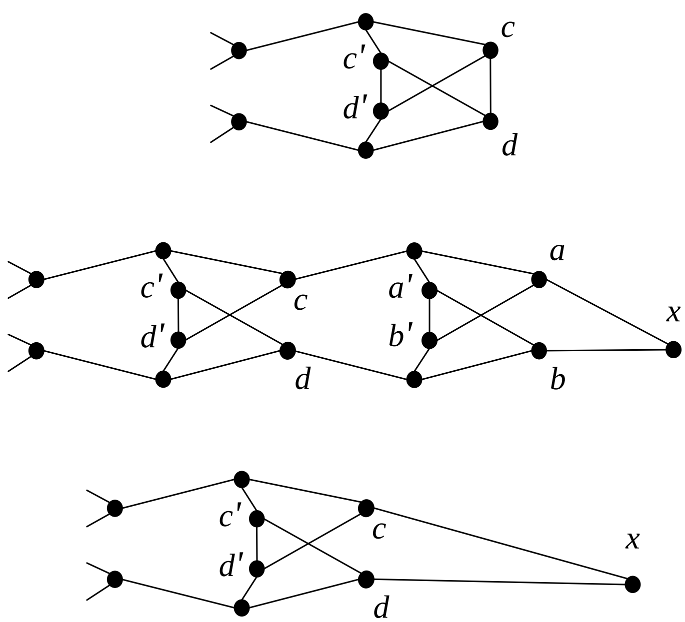

Figure 9: The graph G' at the bottom has the same twins as G.

distinct (as G has no cutpoint other than x and y); they must be nonadjacent (otherwise they would be twins of c' and d'); and they must not share a neighbor (as G has no cutpoint other than x and y).

But then G has the structure indicated in the second row of Figure 9, and clearly the graph G' indicated at the bottom of the figure has the same twin vertices G has. This graph is a circle graph, because it can be obtained from G as follows. First, delete the vertices a' and b'. There are now two unlabeled vertices of degree 2; one of them has a and c as neighbors and the other has b and d as neighbors. Take the local complements with respect to these two unlabeled vertices, and then delete them. Now, take the local complements with respect to a and b; after that, delete a and b.

We have verified our claim: if no twins of G_x are twins in G, then the minimality of G is contradicted. As G does not have two pairs of twins, the claim implies that no twins of G_y are twins in G. But if y is not incident on a 3-circuit then the argument just given contradicts the minimality of G, and if y is incident on a 3-circuit then the argument of Proposition 12 contradicts the minimality of G. ■

3.2 A minimal counterexample is 3-connected

Suppose G is a cubic circle graph that does not have two pairs of twin vertices, and is of the smallest order for such a graph. We have seen that G must be 2-connected.

Suppose $x \neq y$ and $\{x, y\}$ is a minimal vertex cut of G. Each of x, y is of degree 3 in G, so for each of them there is a component of $G - x - y$ connected to that vertex by only one edge.

Case 1. Suppose x and y are neighbors. Each then has only one neighbor in each component of $G - x - y$. If the neighbors in one component are adjacent to each other, then they form a vertex cut with the same properties as $\{x, y\}$. We may assume that those two were originally labeled x and y, and repeat this relabeling process as many times as possible. We may do the same thing with respect to the other component of $G - x - y$, ultimately obtaining the picture of G indicated on the left-hand side of Figure 10, in which neither a and b nor a' and b' are neighbors. Let H and H' be the smaller graphs obtained from G as indicated on the right-hand side of the figure. Then H and H' are both cubic graphs. Moreover, both are circle graphs, as they can be obtained from G using local complementations and vertex deletions; to obtain H, for instance, we delete all vertices outside C except for x and y, perform local complementations at x and y, and then delete x and y.

As G does not have two pairs of twin vertices, one of H, H' must have the property that all of its pairs of twin vertices are disrupted in G. We presume that H has this property. The minimality of G implies that G has vertices c and d that are twins of a and b (respectively) in H. If a and c are adjacent then $N_G(a) = \{c, d, x\}$ and $N_G(c) = \{a, b, d\}$. It cannot be that b and d are adjacent too, for if they were then we would have $N_G(b) = \{c, d, y\}$ and $N_G(d) = \{a, b, c\}$, implying that c and d are adjacent twins in G. Consequently b and d share

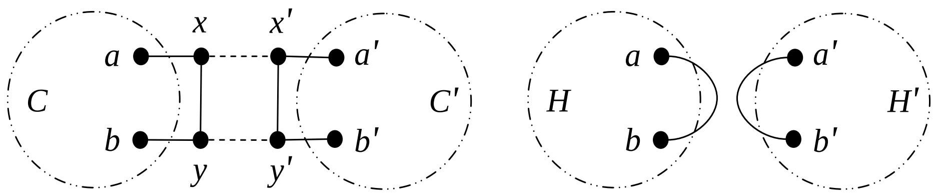

Figure 10: G, H and H' in case 1.

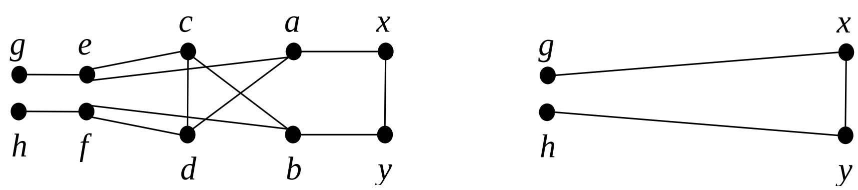

Figure 11: The minimality of G is contradicted in case 1.

a neighbor $e \notin \{a, b, c, d, x, y\}$, and we have $N_G(b) = \{c, e, y\}$ and $N_G(d) = \{a, c, e\}$. But then e is a cutpoint, separating $\{a, b, c, d\}$ from the rest of C. As G has no cutpoint, we conclude that a and c are not adjacent; similarly, b and d are not adjacent.

It follows that G has vertices e and f such that $N_G(a) = \{d, e, x\}$, $N_G(b) = \{c, f, y\}$, $N_G(c) = \{b, d, e\}$ and $N_G(d) = \{a, c, f\}$. As e and f are of degree 3 in G, they must be distinct. If e and f were adjacent, then $\{c, f\}$ and $\{d, e\}$ would be twin pairs in G; hence e and f are not adjacent. Let g and h be the third neighbors of e and f, respectively; g and h must be distinct as G has no cutpoint. The situation in G is pictured on the left in Figure 11. Let G' be the smaller graph obtained from G as indicated on the right in Figure 11. Notice that whether or not g and h are adjacent, neither can have a twin in either G or G'; clearly then all twins of G' are also twins of G, so G' does not have two pairs of twins. Also, G' is a circle graph, because it can be obtained from G by first deleting c and d, and then performing local complementations and vertex deletions at a, b, e and f. But this contradicts the minimality of G.

Case 2. Suppose now that x and y are not neighbors. We denote by C the component of $G - x - y$ that is connected to y by only one edge. We claim that we may suppose without loss of generality that C is connected to x by two edges, as indicated on the left-hand side of Figure 12. Suppose instead that C is connected to x by only one edge; then x has only one neighbor in C, x' say. Necessarily x' is not the neighbor of y in C; if it were, it would be a cutpoint. It follows that $\{x', y\}$ is a minimal vertex cut in G, the induced subgraph $G[V(C) - x']$ is a component of $G - x' - y$, and x' is attached to this component by two edges while y is attached to this component by only one

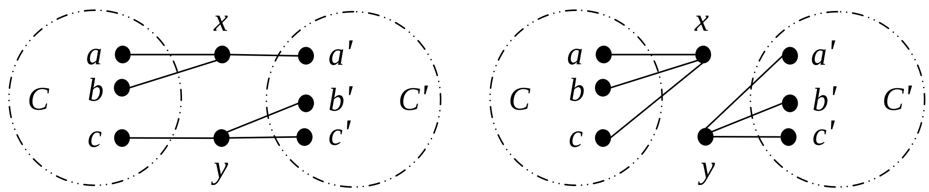

Figure 12: G, H and H' in case 2.

edge.

Having verified our claim, we presume that two edges connect x to C. If x is adjacent to the neighbor of y in C, then x and this neighbor constitute a 2-element vertex cut of G, and we may apply the argument of case 1. The argument of case 1 applies also if x and y share a neighbor in C', so we may proceed with the assumption that x and y do not share a neighbor. That is, the vertices denoted a, b, c, a', b' and c' in Figure 12 are all distinct.

Let H and H' be the two smaller graphs indicated in Figure 12. We claim that they are circle graphs. It is enough to explain why H is a circle graph. Choose a shortest path from a' to one of b', c' in C'. (N.b. Such a path must exist as a' is not a cutpoint of G.) Let H^* be the graph obtained from G by deleting all vertices of C' that do not lie on this path. Then H^* consists of the induced subgraph of G with vertex set $V(C) \cup \{x\}$, and a path of degree-2 vertices connecting x to c. Each of these degree-2 vertices may be removed, by performing a local complementation and a vertex deletion. The result is H.

As H and H' are cubic circle graphs smaller than G, each of them has two disjoint pairs of twin vertices. As G does not have two disjoint pairs of twin vertices, there must be one of H, H' for which all pairs of twins are disrupted in G. We presume that no twins of H are twins in G.

The only vertices of H with different neighbors in G and H are c and x. Consequently, it must be that c and x are elements of two disjoint pairs of twins in H, $\{x, x'\}$ and $\{c, c'\}$. Then c' is a neighbor of x, so c' is one of a, b; we may presume that $c' = b$.

Suppose x' is an adjacent twin of x; then it must be one of a, b. (N.b. This situation is not illustrated in a figure.) As the pairs $\{x, x'\}$ and $\{c, c'\}$ are disjoint, it must be that $x' = a$. Then b and c are both neighbors of a. If b and c were adjacent, it would follow that a and b are adjacent twins in G, contrary to the hypothesis that no twin vertices of H are twin vertices of G. We conclude that b and c are nonadjacent twins in H. Consequently there is a vertex z of C such that $N_H(b) = N_H(c) = \{a, x, z\}$. This cannot happen, though, because every path from x to the third neighbor of z would pass through z, i.e., z would be a cutpoint of G.

Suppose now that x' is a nonadjacent twin of x in H; then $N_G(x') = N_H(x) = \{a, b, c\}$. If a and b were neighbors they would be adjacent twins in G, contrary to hypothesis; so a and b are not neighbors. The situation in G

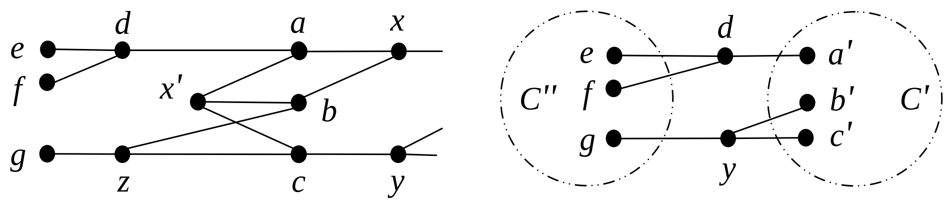

Figure 13: The minimality of G is contradicted in case 2.

is pictured on the left-hand side of Figure 13.

Consider the graph G' indicated on the right-hand side of Figure 13. G' is a circle graph because we can obtain it from G by first deleting b and x', and then performing local complementations and vertex deletions at a, c, x and z. As G' is smaller than G, it has two disjoint pairs of twin vertices. But clearly every pair of twins in G' yields a pair of twins in G, and this contradicts the minimality of G.

As we have reached contradictions in both cases, we conclude that a minimal counterexample to Theorem 5 must be 3-connected.

3.3 No counterexample is 3-connected

The argument in this subsection is quite different from the earlier arguments, as it is focused on the properties of double occurrence words. We begin with the following special case of Bouchet's theorem [2] that two double occurrence words with the same prime interlacement graph must be cyclically equivalent.

Proposition 14 *[2] Suppose $c > 4$ and C_c is the cycle graph with vertices $v_1, ..., v_c$ (in order). Then up to cyclic equivalence, the only double occurrence word with interlacement graph C_c is this:*

$$v_1 v_c v_2 v_1 v_3 v_2 v_4 ... v_{c-1} v_{c-2} v_c v_{c-1} \tag{1}$$

Although C_3 and C_4 are not prime, a version of Proposition 14 applies to them too.

Proposition 15 *Suppose $c \in \{3, 4\}$ and W is a double occurrence word with $\mathcal{I}(W) = C_c$. Then the vertices of C_c may be indexed in such a way that $v_1, ..., v_c$ appear in this order on the cycle, and W is cyclically equivalent to $v_1 v_3 v_2 v_1 v_3 v_2$ (if $c = 3$) or $v_1 v_4 v_2 v_1 v_3 v_2 v_4 v_3$ (if $c = 4$).*

Proof. Suppose first that $W = w_1 w_2 w_3 w_4 w_5 w_6$ has $\mathcal{I}(W) = C_3$. If w_1, w_2 and w_3 are not distinct letters then one of them has degree ≤ 1 in the interlacement graph, a contradiction. If w_4 is not the same letter as w_1 then the degree of w_4 in the interlacement graph is ≤ 1; hence $w_1 w_2 w_3 w_4 = v_1 v_2 v_3 v_1$.

If w_5 is v_3 then v_3 does not neighbor v_2 in $\mathcal{I}(W)$, a contradiction; hence $w_1w_2w_3w_4w_5 = v_1v_2v_3v_1v_2$ and of course $w_6 = v_3$ as that is the only remaining possibility.

Now suppose $W = w_1w_2w_3w_4w_5w_6w_7w_8$ has $\mathcal{I}(W) = C_4$. Let v_1 denote the vertex corresponding to w_1. Suppose the vertex corresponding to w_2 is adjacent to v_1 in C_4, and denote the corresponding vertex v_4. The vertex corresponding to w_3 cannot be either v_1 or v_4, for if it were then its degree in $\mathcal{I}(W)$ would be < 2. Call this vertex v_2. If v_2 is not adjacent to v_1 then W must be of the form $v_1v_4v_2...v_2...v_1...$ But then it is impossible to place the second appearance of v_4 so as to interlace both v_1 and v_2. Consequently v_2 is adjacent to v_1. The vertex corresponding to w_4 cannot be either v_2 or v_4, for if it were then its degree would be < 2. If it is the remaining vertex v_3 then as v_1 and v_3 are not adjacent in C_4, W is of the form $v_1v_4v_2v_3...v_3...v_1...$ and it is impossible to locate the second appearance of v_2 so as to interlace both v_1 and v_3. Consequently w_4 is the second appearance of v_1, and W is of the form $v_1v_4v_2v_1w_5w_6w_7w_8$. The remaining vertex v_3 is of degree 2 in $\mathcal{I}(W)$, and this can only happen if w_5 and w_8 are its two appearances. Necessarily then w_6 and w_7 are v_2 and v_4; as they are not neighbors in $\mathcal{I}(W)$, W must be of the form $v_1v_4v_2v_1v_3v_2v_4v_3$.

It remains to consider the possibility that $W = w_1w_2w_3w_4w_5w_6w_7w_8$ has $\mathcal{I}(W) = C_4$, v_1 denotes the vertex corresponding to w_1, and the vertex corresponding to w_2 is not adjacent to v_1 in C_4. In this case we observe that the vertex corresponding to w_8 cannot be the vertex corresponding to either w_1 or w_2; if it were its degree in $\mathcal{I}(W)$ would be < 2. Consequently w_8 is a neighbor of v_1, and we may apply the argument of the preceding paragraph to the cyclically equivalent word $w_1w_8w_7w_6w_5w_4w_3w_2$. ∎

Note that a cubic graph cannot be a forest, as it has no vertex of degree 1. Consequently every cubic graph has circuits. Suppose W is a double occurrence word whose interlacement graph is $\mathcal{I}(W) = G$, a cubic circle graph. Let C be a circuit in G, of minimal length $c \geq 3$. If we remove from W all occurrences of vertices that do not appear on C, we must obtain a subword W' whose interlacement graph is C. (Note that the minimality of c guarantees that C has no chord in G.) Propositions 14 and 15 tell us that we may index the vertices that appear on C as v_1, ..., v_c, in order of their appearance, and W' will be cyclically equivalent to (1). Consequently, we may assume that W is of the form

$$v_1W_1v_cW_2v_2W_3v_1W_4v_3W_5v_2W_6v_4W_7v_3...v_{c-2}W_{2c-2}v_cW_{2c-1}v_{c-1}W_{2c}. \qquad (2)$$

When we reference this description of W we will consider the index of W_i modulo $2c$, so that $W_0 = W_{2c}$, $W_1 = W_{2c+1}$, etc.

Observe that if v appears in two non-consecutive W_i then v must neighbor at least two of $v_1, ..., v_c$. For instance, a vertex that appears once in W_2 and once in W_6 neighbors v_1 and v_3. On the other hand, a vertex that appears once in each of two consecutive W_i neighbors exactly one of $v_1, ..., v_c$, and a vertex that appears twice in the same W_i does not neighbor any of $v_1, ..., v_c$.

Proposition 16 *Let G be a cubic circle graph that does not have two pairs of*

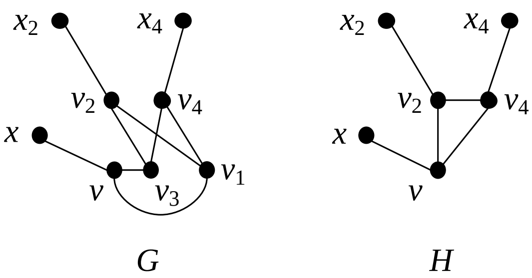

Figure 14: The last case of the claim.

twin vertices. Then G is not 3-connected.

Proof. Suppose instead that G is a 3-connected, cubic circle graph, which does not have two pairs of twin vertices. We may presume that G is of the smallest possible order for such a graph. Let W be a double occurrence word of the form (2), with $\mathcal{I}(W) = G$.

Claim If $v \notin \{v_1, ..., v_c\}$ then either v appears twice in the same subword W_i, or v appears once in each of two consecutive subwords W_i and W_{i+1}.

Suppose the claim is false; then some $v \notin \{v_1, ..., v_c\}$ neighbors more than one of $v_1, ..., v_c$. Cyclically permuting indices if necessary we may presume that v neighbors v_1 and some other v_j.

If v neighbors v_j, $j > 3$, then $v_1, v, v_j, ..., v_c$ is a closed walk in G, of length $< c$. This contradicts the choice of c. If v neighbors v_2 then $\{v, v_1, v_2\}$ is a circuit in G, so $c = 3$. But then v_1 and v_2 both neighbor v, and also v_1 and v_2 both neighbor v_3. As v_1 and v_2 are neighbors of degree 3, it follows that $\{v, v_3\}$ is a vertex cut, which separates $\{v_1, v_2\}$ from the rest of G. As G is 3-connected, there must be no "rest of G" – i.e., v, v_1, v_2 and v_3 are all the vertices G has. But then G is a 4-clique, contradicting the hypothesis that G does not have two disjoint pairs of twin vertices.

Suppose v neighbors v_1 and v_3. Then v, v_1, v_2, v_3 is a closed walk in G, so $c \leq 4$. If $c = 3$ then v_1, v_2, v_3 and v_1, v, v_3 are both 3-circuits in G, so $\{v, v_2\}$ is a vertex cut that separates $\{v_1, v_3\}$ from the rest of G. Again, this contradicts either the hypothesis that G is 3-connected or the hypothesis that G does not have two disjoint pairs of twin vertices.

We conclude that $c = 4$. Then v_1, v_2, v_3, v_4 is a circuit of G, so v_1 and v_3 both neighbor v, v_2 and v_4. Consequently, v_1 and v_3 are nonadjacent twins. No two of v, v_2 and v_4 may share another neighbor; for if they were to share another neighbor then they would be twins, and G does not have two pairs of twins. Let x, x_2 and x_4 be the third neighbors of v, v_2 and v_4 respectively, as on the left-hand side of Figure 14.

Let H be the graph obtained by replacing the pictured portion of G with the smaller subgraph indicated on the right-hand side of Figure 14. Then H

is obtained from G in three steps: delete v_1, take the local complement with respect to v_3, and delete v_3. Consequently H is a circle graph. Clearly H is also cubic and 3-connected.

The minimality of G assures us that H has two pairs of twin vertices. The six indicated vertices of H are all distinct, so no two of them are twins (as their neighborhoods are distinct). Any twin vertices outside the pictured portion of H are also twins in G, though, and this implies that G has three pairs of twin vertices, contradicting the hypothesis that it does not even have two pairs. The contradiction verifies the claim.

As G is cubic and C is a chordless circuit, each v_j has precisely one neighbor u_j that does not appear on C. The claim tells us that u_j appears in two consecutive subwords W_i and W_{i+1}. As u_j neighbors v_j, $W_i v_j W_{i+1} v_k$ must be a subword of W, for some k. Notice that as W_i and W_{i+1} do not mention any of v_1, ..., v_c, u_j does not neighbor any of v_1, ..., v_c other than v_j; this holds for every j, so u_1, ..., u_c are pairwise distinct.

If W_{i+1} contains any vertex v other than u_j or u_k, then v appears twice in W_{i+1}, so every walk from v to v_j in G must pass through u_j or u_k. As G is 3-connected, it follows that there is no such v. The same argument applies to every W_i, as the claim implies that no more than two of u_1, ..., u_c appear in any one W_i. We conclude that $V(G) = \{u_1, ..., u_c, v_1, ..., v_c\}$.

After reversing or cyclically permuting W if necessary, we may presume that u_1 appears in W_3 and W_4. The degree of u_1 is 3, so the subword $v_2 W_3 v_1 W_4 v_3$ of W must be $v_2 u_1 u_2 v_1 u_3 u_1 v_3$. The degree of u_2 is also 3, so the subword $v_c W_2 v_2 W_3 v_1 W_4 v_3$ must be $v_c u_2 u_c v_2 u_1 u_2 v_1 u_3 u_1 v_3$. Notice that u_2 cannot appear in W_5, as it appears in W_2 and W_3; consequently the subword $W_4 v_3 W_5 v_2$ of W must be $u_3 u_1 v_3 u_3 v_2$. But then the degree of u_3 is 2, contradicting the hypothesis that G is 3-regular. ■

4 Corollary 7

In this section we derive Corollary 7 from Theorem 5.

Suppose G is a cubic circle graph, which is 3-connected. If G has a pair of adjacent twins, v and w, then they share two neighbors, x and y. If z is any other vertex of G then every path from v to z in G must pass through x or y; as G is 3-connected, this cannot be the case. Consequently G has no other vertex, i.e., $V(G) = \{v, w, x, y\}$. As G is a cubic graph, $G \cong K_4$.

If G has no pair of adjacent twins then Theorem 5 tells us that G has two disjoint pairs of nonadjacent twins, $\{v, v'\}$ and $\{w, w'\}$. Suppose v and w are neighbors; then v and v' are neighbors of w and w'. Let the third neighbor of v and v' be x, and let the third neighbor of w and w' be y. If $V(G) \neq \{v, v', w, w', x, y\}$, then every path from one of v, v', w, w' to a vertex outside $\{v, v', w, w', x, y\}$ passes through x or y. As G is 3-connected, this cannot be the case; we conclude that $V(G) = \{v, v', w, w', x, y\}$ and hence $G \cong K_{3,3}$.

Suppose now that v and w are not neighbors; we claim that this is impossible. The graph G is 3-connected, so Menger's theorem tells us that there are three

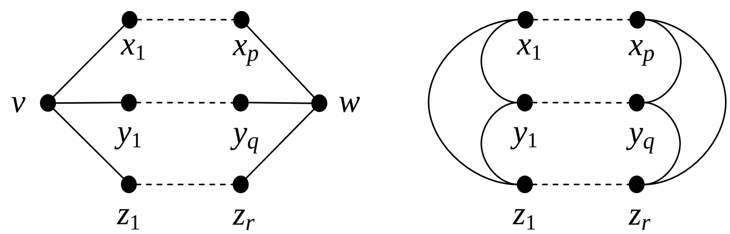

Figure 15: The graphs H (on the left) and H', H'' (on the right).

internally vertex-disjoint paths from v to w. We may presume that no edge of G connects two non-consecutive vertices of any of the three paths; for if there is such an edge we may use it to shorten that path. Let H be the full subgraph of G induced by the vertices on these three paths, including v and w. Then H is a circle graph. Let the three paths be v, x_1, ..., x_p, w; v, y_1, ..., y_q, w; and v, z_1, ..., z_r, w. H is pictured on the left in Figure 15.

Note that as G is 3-regular, it must be that $x_1 \neq x_p$; for x_1 is adjacent to both v and v', and x_p is adjacent to both w and w'. The same argument tells us that $y_1 \neq y_q$ and $z_1 \neq z_r$. As v' and w' are not vertices of H, x_1, y_1, z_1, x_p, y_q and z_r are all of degree 2 in H. Consequently x_1, y_1, z_1, x_p, y_q and z_r are all of degree 3 in the graph H' obtained from H by performing local complementations and vertex deletions at v and w. If any vertex of H' is of degree 2, we may remove it by performing a local complementation and then a vertex deletion. The resulting graph H'' differs from H' in that some indices may not appear on the paths x_1, ..., x_p; y_1, ..., y_q; and z_1, ..., z_r. But each path will still involve at least two distinct vertices, so no two vertices of H'' will be twins.

As H'' is a cubic circle graph, Theorem 5 verifies the claim that this situation is impossible.

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